ON EQUIVALENCY OF VARIOUS GEOMETRIC STRUCTURES

ABSOS ALI SHAIKH* AND HARADHAN KUNDU

Abstract. In the literature we see that after introducing a geometric structure by imposing some restrictions on Riemann-Christoffel curvature tensor, the same type structure given by imposing same restriction on other curvature tensors being studied. The main object of the present paper is to study the equivalency of various geometric structures obtained by same restriction imposing on different curvature tensors. In this purpose we present a tensor by combining Riemann-Christoffel curvature tensor, Ricci tensor, the metric tensor and scalar curvature which describe various curvature tensors as its particular cases.

1. Introduction

Let M be a semi-Riemannian manifold of dimension $n \geq 3$, endowed with the semi-Riemannian metric g with signature $(p, n-p)$, $0 \leq p \leq n$. If (i) $p = 0$ or $p = n$; (ii) $p = 1$ or $p = n-1$, then M is said to be a (i) Riemannian; (ii) Lorentzian manifold respectively. Let ∇, R, S and r be the Levi-Civita connection, Riemannian-Christoffel curvature tensor, Ricci tensor and scalar curvature of M respectively. All the manifold considered here are assumed to be smooth and connected. We note that any two 1-dimensional semi-Riemannian manifolds are locally isometric, and an 1-dimensional semi-Riemannian manifold is a void field. Also for $n = 2$, the notions of above three curvatures are equivalent. Hence throughout the study we will confined ourselves with a semi-Riemannian manifold M of dimension $n \geq 3$. In the study of differential geometry there are various theme of research to derive the geometric properties of a semi-Riemannian manifold. Among others symmetry plays an important role in the study of differential geometry of a semi-Riemannian manifold.

As a generalization of manifolds of constant curvature, the notion of local symmetry was introduced by Cartan [4] with a full classification for the Riemann case. A full classification of such notion was given by Cahen and Parker ([2], [3]) for indefinite case. The manifold M is said to be locally symmetric if its local geodesic symmetries are isometry and M is said to be globally symmetric if its geodesic symmetries are extendible to the whole of M. Every locally symmetric manifold is globally symmetric but not conversely. For instance, every compact Riemann surface of genus> 1 endowed with its usual metric of constant curvature (-1) is locally symmetric but not globally symmetric. We note that the famous Cartan-Ambrose-Hicks theorem implies that M is locally symmetric if and

[0*] Corresponding author.

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only if $\nabla R = 0$, and any simply connected complete locally symmetric manifold is globally symmetric. During the last eight decades the notion of local symmetry has been weakened by many ways in different directions by "imposing the restriction on R by means of first order or higher order covariant derivatives" and we will call them simply "curvature restriction on R". And in the literature there are many geometric structures arise in this way. For example, recurrency, semisymmetry, pseudosymmetry etc. are geometric structures obtained by imposing the restriction on R with first and second order covariant differential. All these geometric structures are studied by many geometers with their classification, existence and applications. In differential geometry there are various curvature tensors arise as an invariant of different transformations, e.g., projective (P), conformal (C), concircular (W), conharmonic (K) curvature tensors etc. In the literature there are many papers where the same curvature restriction is studied with other curvature tensors which are either meaningless or redundant due to their equivalency, for example, $\nabla R = 0$ and $\nabla P = 0$ are equivalent [28]. The main object of this paper is to prove the equivalency of various geometric structures obtained by the same curvature restriction on different curvature tensors. For this purpose we present a (0,4) tensor B by combining Riemann-Christoffel curvature tensor, Ricci tensor, the metric tensor and scalar curvature which describes various curvature tensors as its particular cases. The tensor of the form like B are said to be B-tensors and the set of all B-tensors will be denoted by $\mathscr{B}$. We classify the set $\mathscr{B}$ with respect to contraction such that in each class, several geometric structures obtained by the same curvature restriction are equivalent.

We are interested mainly on those geometric structures which are obtained by imposing restrictions as some operators on various curvature tensors. The work on this paper assembled such geometric restrictions and we classify these geometric restrictions with respect to their linearity and commutivity with contraction and study the results for each class of restrictions together. On the basis of this study we can say that a specific curvature restriction provides us how many different geometric structures arise due to different curvature tensors.

In section 2 we present the B-tensor and showed various curvature tensors which are introduced already, are particular cases of it. Section 3 deals with preliminaries. In section 4 we classify the geometric structures (actually generalized or extended or weaker structures of symmetry defined by Cartan) and give the definitions of various geometric structures. Section 5 is concerned with basic well known results and some basic properties of the tensors B. In section 6 we classify the B-tensor and calculate the main results on equivalency of structures. Finally in last section we make conclusion of the whole work.

2. B-tensor

Let M be an $n(\geq 3)$-dimensional connected semi-Riemannian manifold equipped with the metric g. We denote by ∇, R, S, r, the Levi-Civita connection, the Riemann-Christoffel curvature tensor, Ricci tensor and scalar curvature of M respectively. We define a $(0,4)$ tensor B given by

$$\begin{aligned}
B(X_1,X_2,X_3,X_4) &= a_0R(X_1,X_2,X_3,X_4) + a_1R(X_1,X_3,X_2,X_4) \\
&+a_2S(X_2,X_3)g(X_1,X_4) + a_3S(X_1,X_3)g(X_2,X_4) + a_4S(X_1,X_2)g(X_3,X_4) \\
&+a_5S(X_1,X_4)g(X_2,X_3) + a_6S(X_2,X_4)g(X_1,X_3) + a_7S(X_3,X_4)g(X_1,X_2) \\
&+r\big[a_8g(X_1,X_4)g(X_2,X_3) + a_9g(X_1,X_3)g(X_2,X_4) + a_{10}g(X_1,X_2)g(X_3,X_4)\big],
\end{aligned} \tag{2.1}$$

where a_i's are scalars on M and $X_1, X_2, Y_1, Y_2 \in \chi(M)$, the Lie algebra of all smooth vector fields on M. Now we see that B reduce to various curvature tensors such as (i) Riemann-Christoffel curvature tensor R, (ii) Weyl conformal curvature tensor C, (iii) projective curvature tensor P, (iv) concircular curvature tensor W [37], (v) conharmonic curvature tensor K [20], (vi) quasi conformal curvature tensor C^* [38], (vii) pseudo projective curvature tensor P^* [27], (viii) $\mathcal{M}$-projective curvature tensor [26], (ix) $\mathcal{W}_i$-curvature tensor, $i = 1, 2, ..., 9$ ([24], [25], [26]) and (x) $\mathcal{W}_i^*$-curvature tensor, $i = 1, 2, ..., 9$ ([26]) for different value of a_i's, given by:

Tensor	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
R	1	0	0	0	0	0	0	0	0	0	0
C	1	0	$-\frac{1}{n-2}$	$\frac{1}{n-2}$	0	$-\frac{1}{n-2}$	$\frac{1}{n-2}$	0	$\frac{1}{(n-1)(n-2)}$	$-\frac{1}{(n-1)(n-2)}$	0
P	1	0	$-\frac{1}{n-1}$	0	0	0	$\frac{1}{n-1}$	0	0	0	0
W	1	0	0	0	0	0	0	0	$-\frac{1}{n(n-1)}$	$\frac{1}{n(n-1)}$	0
K	1	0	$-\frac{1}{n-2}$	$\frac{1}{n-2}$	0	$-\frac{1}{n-2}$	$\frac{1}{n-2}$	0	0	0	0
C^*	a_0	0	a_2	$-a_2$	0	a_2	$-a_2$	0	$-\frac{1}{n}\left(\frac{a_0}{n-1}+2a_2\right)$	$\frac{1}{n}\left(\frac{a_0}{n-1}+2a_2\right)$	0
P^*	a_0	0	a_2	$-a_2$	0	0	0	0	$-\frac{1}{n}\left(\frac{a_0}{n-1}+a_2\right)$	$\frac{1}{n}\left(\frac{a_0}{n-1}+a_2\right)$	0
$\mathcal{M}$	1	0	$-\frac{1}{2(n-1)}$	$\frac{1}{2(n-1)}$	0	$-\frac{1}{2(n-1)}$	$\frac{1}{2(n-1)}$	0	0	0	0
$\mathcal{W}_0$	1	0	$-\frac{1}{n-1}$	0	0	0	$\frac{1}{n-1}$	0	0	0	0
$\mathcal{W}_0^*$	1	0	$\frac{1}{n-1}$	0	0	0	$-\frac{1}{n-1}$	0	0	0	0
$\mathcal{W}_1$	1	0	$-\frac{1}{n-1}$	$\frac{1}{n-1}$	0	0	0	0	0	0	0
$\mathcal{W}_1^*$	1	0	$\frac{1}{n-1}$	$-\frac{1}{n-1}$	0	0	0	0	0	0	0
$\mathcal{W}_2$	1	0	0	0	0	$-\frac{1}{n-1}$	$\frac{1}{n-1}$	0	0	0	0
$\mathcal{W}_2^*$	1	0	0	0	0	$\frac{1}{n-1}$	$-\frac{1}{n-1}$	0	0	0	0
$\mathcal{W}_3$	1	0	0	$-\frac{1}{n-1}$	0	$\frac{1}{n-1}$	0	0	0	0	0
$\mathcal{W}_3^*$	1	0	0	$\frac{1}{n-1}$	0	$-\frac{1}{n-1}$	0	0	0	0	0
$\mathcal{W}_4$	1	0	0	0	0	0	$-\frac{1}{n-1}$	$\frac{1}{n-1}$	0	0	0
$\mathcal{W}_4^*$	1	0	0	0	0	0	$\frac{1}{n-1}$	$-\frac{1}{n-1}$	0	0	0
$\mathcal{W}_5$	1	0	0	$-\frac{1}{n-1}$	0	0	$\frac{1}{n-1}$	0	0	0	0
$\mathcal{W}_5^*$	1	0	0	$\frac{1}{n-1}$	0	0	$-\frac{1}{n-1}$	0	0	0	0
$\mathcal{W}_6$	1	0	$-\frac{1}{n-1}$	0	0	0	0	$\frac{1}{n-1}$	0	0	0
$\mathcal{W}_6^*$	1	0	$\frac{1}{n-1}$	0	0	0	0	$-\frac{1}{n-1}$	0	0	0
$\mathcal{W}_7$	1	0	$-\frac{1}{n-1}$	0	0	$\frac{1}{n-1}$	0	0	0	0	0
$\mathcal{W}_7^*$	1	0	$\frac{1}{n-1}$	0	0	$-\frac{1}{n-1}$	0	0	0	0	0
$\mathcal{W}_8$	1	0	$-\frac{1}{n-1}$	0	$\frac{1}{n-1}$	0	0	0	0	0	0
$\mathcal{W}_8^*$	1	0	$\frac{1}{n-1}$	0	$-\frac{1}{n-1}$	0	0	0	0	0	0
$\mathcal{W}_9$	1	0	0	0	$-\frac{1}{n-1}$	$\frac{1}{n-1}$	0	0	0	0	0
$\mathcal{W}_9^*$	1	0	0	0	$\frac{1}{n-1}$	$-\frac{1}{n-1}$	0	0	0	0	0

There may arise some other tensors from the tensor B as its particular cases, which are not introduced so far. The tensors reduced from B given in (2.1) are called B-tensors. We denote the set of all B-tensors by $\mathscr{B}$.

3. **Preliminaries**

Let us now consider a connected semi-Riemannian manifold M of dimension n(≥ 3). Then for two (0, 2) tensors A and E, the Kulkarni-Nomizu product ([8],, [13],[18], [19], [20], [21]) $A \wedge E$ is given

by

$$(A \wedge E)(X_1, X_2, X_3, X_4) = A(X_1, X_4)E(X_2, X_3) + A(X_2, X_3)E(X_1, X_4) - A(X_1, X_3)E(X_2, X_4) - A(X_2, X_4)E(X_1, X_3), \tag{3.1}$$

where $X_1, X_2, Y_1, Y_2 \in \chi(M)$.
A tensor D of type (1,3) on M is said to be generalized curvature tensor ([11], [12], [15]), if

$$(i) \quad D(X_1, X_2)X_3 + D(X_2, X_1)X_3 = 0,$$
$$(ii) \quad D(X_1, X_2, X_3, X_4) = D(X_3, X_4, X_1, X_2),$$
$$(iii) \quad D(X_1, X_2)X_3 + D(X_2, X_3)X_1 + D(X_3, X_1)X_2 = 0,$$

where $D(X_1, X_2, X_3, X_4) = g(D(X_1, X_2)X_3, X_4)$, for all X_1, X_2, X_3, X_4. Here we denote the same symbol D for both generalized curvature tensor of type (1,3) and (0,4). Moreover if D satisfies the second Bianchi identity i.e.,

$$(\nabla_{X_1} D)(X_2, X_3)X_4 + (\nabla_{X_2} D)(X_3, X_1)X_4 + (\nabla_{X_3} D)(X_1, X_2)X_4 = 0,$$

then D is called a proper generalized curvature tensor. We note that a linear combination of generalized curvature tensors over $C^\infty(M)$, the set of all smooth functions on M, is again a generalized curvature tensor but it is not true for proper generalized curvature tensors. However, if the linear combination is taken over $\mathbb{R}$, then it is true.

Now for any (1,3) tensor D (not necessarily generalized curvature tensor) and given two vector fields $X, Y \in \chi(M)$, one can define an endomorphism $\mathcal{D}(X, Y)$ by

$$\mathcal{D}(X, Y)(Z) = D(X, Y)Z, \quad \forall Z \in \chi(M).$$

Again, if $X, Y \in \chi(M)$ then for a (0,2) tensor A, one can define two endomorphisms $\mathcal{A}$ and $X \wedge_A Y$, by ([11], [12], [15])

$$g(\mathcal{A}(X), Y) = A(X, Y),$$
$$(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y, \ \forall \ Z \in \chi(M).$$

Now for a $(0, k)$-tensor T, $k \geq 1$, and an endomorphism $\mathcal{H}$, one can operate $\mathcal{H}$ on T to produce the tensor $\mathcal{H}T$ given by ([11], [12], [15])

$$(\mathcal{H}T)(X_1, X_2, \cdots, X_k) = -T(\mathcal{H}X_1, X_2, \cdots, X_k) - \cdots - T(X_1, X_2, \cdots, \mathcal{H}X_k).$$

We consider that operation of $\mathcal{H}$ on a scalar is zero. In particular, $\mathcal{H}$ may be $\mathcal{D}(X, Y)$, $X \wedge_A Y$, $\mathcal{A}$ etc. For $\mathcal{H} = \mathcal{D}(X, Y)$, we write $\mathcal{H}T = \mathcal{D}(X, Y)T$ as $D \cdot T$. We also note that $(X \wedge_A Y) \cdot T$ is

written as $Q(A,T)$ and thus is given by ([11], [12], [15], [34])

$$\begin{aligned}
Q(A,T)(X_1,X_2, \cdots, X_k,X,Y) &= ((X \wedge_A Y)\cdot T)(X_1,X_2,\cdots,X_k)\\
&= -T((X \wedge_A Y)X_1,X_2,\cdots,X_k) - \cdots - T(X_1,X_2,\cdots,(X \wedge_A Y)X_k)\\
&= A(X,X_1)T(Y,X_2,\cdots,X_k) + \cdots + A(X,X_k)T(X_1,X_2,\cdots,Y)\\
&\quad -A(Y,X_1)T(X,X_2,\cdots,X_k) - \cdots - A(Y,X_k)T(X_1,X_2,\cdots,X),
\end{aligned}$$

where $X,Y,X_i \in \chi(M)$, $i = 1,2,\cdots,k$.
For an 1-form Π and a vector field X on M, we can define an endomorphism Π_X as

$$\Pi_X(X_1) = \Pi(X_1)X, \ \forall X_1 \in \chi(M).$$

Then we can define Π_X as an operation on a $(0,k)$ tensor field T as follows:

$$\begin{aligned}
&(\Pi_X \cdot T)(X_1,X_2,\cdots,X_k)\\
&= -T(\Pi_X(X_1),X_2,\cdots,X_k) - \cdots - T(X_1,X_2,\cdots,\Pi_X(X_k)),\\
&= -\Pi(X_1)T(X,X_2,\cdots,X_k) - \Pi(X_2)T(X_1,X,\cdots,X_k) - \cdots - \Pi(X_k)T(X_1,X_2,\cdots,X),
\end{aligned}$$

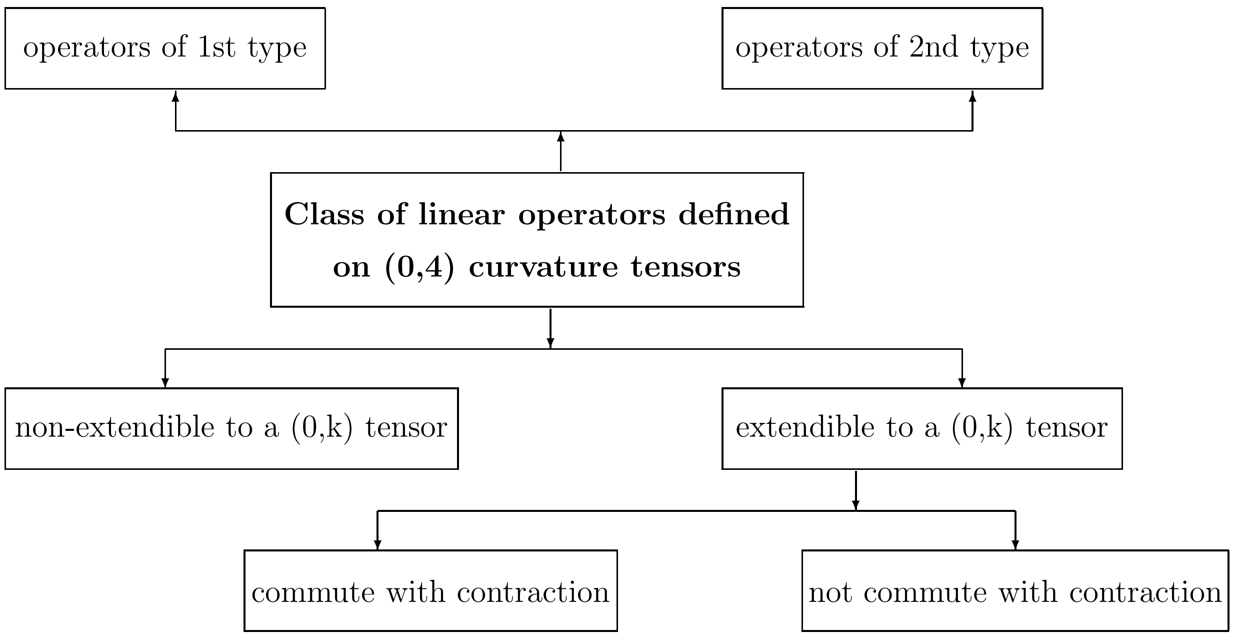
$\forall X, X_i \in \chi(M), i = 1,2,\cdots,k.$

4. **Some geometric structures defined by curvature related operators**

Now we discuss about the geometric structures arise by the curvature restrictions of a semi-Riemannian manifold. We are mainly interested on those geometric structures which are obtained by some curvature restrictions imposed on B-tensors by means of some operators, e.g., symmetry, recurrency, pseudosymmetry etc. These operators are linear over $\mathbb{R}$ and may or may not be linear over $C^\infty(M)$ and thus called as $\mathbb{R}$-linear operators or simply linear operators. The linear operators which are not linear over $C^\infty(M)$, said to be *operators of the 1st type* and which are linear over $C^\infty(M)$, said to be *operators of the 2nd type*. Some important 1st type operators are symmetry, recurrency, weakly symmetry (in the sense of Tamassy and Binh) etc. and some important 2nd type operators are semisymmetry, Deszcz pseudosymmetry, Ricci generalized pseudosymmetry etc. For this purpose we consider the set of all tensor fields of order (k,s) as $\mathcal{T}^k_s(M)$ and take $\mathcal{L}$ as such an operator which operates on T denoted as $\mathcal{L}\ T$.

Another classification of such geometrical operators may be given with respect to their extendibility. Actually these operators are imposed on $(0,4)$ curvature tensors but the defining condition of some of them can not be extended to any $(0,k)$ tensor, e.g., symmetry, semisymmetry, weak symmetry (all three types) operators are extendible but weakly generalized recurrency, hyper generalized recurrency operators are not extendible. Again extendible operators are classified into two subclasses,

(i) operators commute with contraction or commutative and (ii) operators not commute with contraction or non-commutative., e.g., symmetry, semisymmetry operators are commutative but weak symmetry operators are non-commutative. Throughout this paper we mean by commutative or non-commutative operator as it commute or not commute with contraction. The tree diagram of the classification of linear operators imposed on (0,4) curvature tensors is given by:



Definition 4.1. [4] *Consider the covariant derivative operator* $\nabla_X : \mathcal{T}^0_k \to \mathcal{T}^0_{k+1}$*. A semi-Riemannian manifold is said to be* T*-symmetric if* $\nabla_X T = 0$*, for all* $X \in \chi(M)$*.*

Obviously this operator is of 1st type and commutative. The condition for T-symmetry is written as $\nabla T = 0$.

Definition 4.2. [36] *Consider the operator* $\kappa_{(X,\Pi)} : \mathcal{T}^0_k \to \mathcal{T}^0_{k+1}$ *defined by* $\kappa_{(X,\Pi)} T = \nabla_X T - \Pi(X) \otimes T$*,* Π *is an 1-form and* $T \in \mathcal{T}^0_k$*. A semi-Riemannian manifold is said to be* T*-recurrent if* $\kappa_{(X,\Pi)} T = 0$ *for all* $X \in \chi(M)$ *and some 1-form* Π*, called the associated 1-form or the 1-form of recurrency.*

Obviously this operator is of 1st type and commutative. The condition for T-recurrency is written as $\nabla T - \Pi \otimes T = 0$ or simply $\kappa T = 0$.

Keeping the properties to be of 1st type and commutative we state some generalization of symmetry operator and recurrency operator which are respectively said to be *symmetric type* operator and *recurrent type* operator. For this purpose we denote the s-th covariant derivative as

$$\nabla_{X_1} \nabla_{X_2} \cdots \nabla_{X_s} = \nabla^s_{X_1 X_2 \cdots X_s}.$$

Now the operator

$$L^s_{X_1 X_2 \cdots X_s} = \sum_{\sigma} \alpha_\sigma \nabla^s_{X_{\sigma(1)} X_{\sigma(2)} \cdots X_{\sigma(s)}}$$

is called a symmetric type operator of order s, where σ is permutation over $\{1,2,...,s\}$ and the sum is taken over the set of all permutations over $\{1,2,...,s\}$ and α_σ's are some scalars not all together zero. A manifold is called T-symmetric type of order s if

$$L^s_{X_1X_2\cdots X_s}T = 0 \quad \forall X_1, X_2, \cdots X_s \in \chi(M). \tag{4.1}$$

The condition for T-symmetry of order s is written as $L^sT = 0$.

Again for some $(0,i)$ tensors Π^i_σ, $i = 1,2,\cdots,s$ and all permutations σ over $\{1,2,...,s\}$ (i.e., Π^0_σ are scalars), the operator

$$\begin{aligned}
\kappa^s_{X_1X_2\cdots X_s} \quad = \quad & \sum_\sigma \Big[\Pi^0_\sigma \nabla^s_{X_{\sigma(1)}X_{\sigma(2)}\cdots X_{\sigma(s)}} \\
& +\Pi^1_\sigma(X_{\sigma(1)})\nabla^{s-1}_{X_{\sigma(2)}X_{\sigma(3)}\cdots X_{\sigma(s)}} \\
& +\Pi^2_\sigma(X_{\sigma(1)}, X_{\sigma(2)})\nabla^{s-2}_{X_{\sigma(3)}X_{\sigma(4)}\cdots X_{\sigma(s)}} \\
& +\cdots \quad \cdots \quad \cdots \quad \cdots \\
& +\Pi^{s-1}_\sigma(X_{\sigma(1)}, X_{\sigma(2)}, \cdots, X_{\sigma(s-1)})\nabla_{X_{\sigma(s)}} \\
& +\Pi^s_\sigma(X_{\sigma(1)}, X_{\sigma(2)}, \cdots, X_{\sigma(s)})\Big]
\end{aligned}$$

is called a recurrent type operator of order s. A manifold is called T-recurrent type of order s if it satisfies

$$\kappa^s_{X_1X_2\cdots X_s}T = 0, \quad \forall X_1, X_2, \cdots X_s \in \chi(M) \text{ and some } i\text{-forms } \Pi^i_\sigma\text{'s}. \tag{4.2}$$

The condition for T-recurrency of order s is simply written as $\kappa^sT = 0$.

Another way to generalize recurrency there are some other geometric structures defined as follows:

Definition 4.3. [16] *Consider the operator $G\kappa_{(X,\Pi,\Phi)} : \mathcal{T}^0_4 \to \mathcal{T}^0_5$ defined by*

$$G\kappa_{(X,\Pi,\Phi)}T = \nabla_X T - \Pi(X) \otimes T - \Phi(X) \otimes G,$$

Π and Φ are 1-forms and T is a $(0,4)$ tensor. A semi-Riemannian manifold is said to be generalized T-recurrent if $G\kappa_{(X,\Pi,\Phi)}T = 0$ for all $X \in \chi(M)$ and some 1-forms Π and Φ, called the associated 1-forms.

Obviously this operator is of 1st type and non-extendible.

Definition 4.4. [30] *Consider the operator $H\kappa_{(X,\Pi,\Phi)} : \mathcal{T}^0_4 \to \mathcal{T}^0_5$ defined by*

$$H\kappa_{(X,\Pi,\Phi)}T = \nabla_X T - \Pi(X) \otimes T - \Phi(X) \otimes g \wedge S,$$

Π *and* Φ *are 1-forms and* T *is a* $(0,4)$ *tensor. A semi-Riemannian manifold is said to be hyper-generalized* T*-recurrent if* $H\kappa_{(X,\Pi,\Phi)}T = 0$ *for all* $X \in \chi(M)$ *and some 1-forms* Π *and* Φ*, called the associated 1-forms.*

Obviously this operator is of 1st type and non-extendible.

Definition 4.5. [32] *Consider the operator* $W\kappa_{(X,\Pi,\Phi)} : \mathcal{T}^0_4 \to \mathcal{T}^0_5$ *defined by*

$$W\kappa_{(X,\Pi,\Phi)}T = \nabla_X T - \Pi(X) \otimes T - \Phi(X) \otimes S \wedge S,$$

Π *and* Φ *are 1-forms and* T *is a* $(0,4)$ *tensor. A semi-Riemannian manifold is said to be weakly generalized* T*-recurrent if* $W\kappa_{(X,\Pi,\Phi)}T = 0$ *for all* $X \in \chi(M)$ *and some 1-forms* Π *and* Φ*, called the associated 1-forms.*

Obviously this operator is of 1st type and non-extendible.

Definition 4.6. [31] *Consider the operator* $Q\kappa_{(X,\Pi,\Phi)} : \mathcal{T}^0_4 \to \mathcal{T}^0_5$ *defined by*

$$Q\kappa_{(X,\Pi,\Phi,\Psi)}T = \nabla_X T - \Pi(X) \otimes T - \Phi(X) \otimes g \wedge [g + \Psi \otimes \Psi],$$

Π*,* Φ *and* Ψ *are 1-forms and* T *is a* $(0,4)$ *tensor. A semi-Riemannian manifold is said to be quasi generalized* T*-recurrent if* $Q\kappa_{(X,\Pi,\Phi,\Psi)}T = 0$ *for all* $X \in \chi(M)$ *and some 1-forms* Π*,* Φ *and* Ψ*, called the associated 1-forms.*

Obviously this operator is of 1st type and non-extendible.

Definition 4.7. *Consider the operator* $S\kappa_{(X,\Pi,\Phi,\Psi,\Theta)} : \mathcal{T}^0_4 \to \mathcal{T}^0_5$ *defined by*

$$S\kappa_{(X,\Pi,\Phi,\Psi,\Theta)}T = \nabla_X T - \Pi(X) \otimes T - \Phi(X) \otimes G - \Psi(X) \otimes g \wedge S - \Theta(X) \otimes S \wedge S,$$

Π*,* Φ*,* Ψ *and* Θ *are 1-forms and* T *is a (0,4) tensor. A semi-Riemannian manifold is said to be super generalized* T*-recurrent if* $S\kappa_{(X,\Pi,\Phi,\Psi,\Theta)}T = 0$ *for all* $X \in \chi(M)$ *and some 1-forms* Π*,* Φ*,* Ψ *and* Θ*, called the associated 1-forms.*

Obviously this operator is of 1st type and non-extendible.

Definition 4.8. [5] *Consider the operator* $CP_{(X,\Pi)} : \mathcal{T}^0_k \to \mathcal{T}^0_{k+1}$ *defined by*

$$CP_{(X,\Pi)}T = \nabla_X T - 2\Pi(X) \otimes T - \Pi_X \cdot T,$$

Π *is an 1-form and* $T \in \mathcal{T}^0_k$*. A semi-Riemannian manifold is said to be Chaki* T*-pseudosymme-tric* [5] *if* $CP_{(X,\Pi)}T = 0$ *for all* $X \in \chi(M)$ *and some 1-form* Π*.*

Obviously this operator is of 1st type and non-commutative.
There are another structures called weakly symmetry (in sense of Tamássy and Binh) which is a generalization of recurrency and Chaki pseudosymmetry. There are three types of weakly symmetry which are given below:

Definition 4.9. *Consider the operator* $W^1_{(X,\overset{\sigma}{\Pi})} : \mathcal{T}^0_k \to \mathcal{T}^0_{k+1}$ *defined by*

$$W^1_{(X,\overset{\sigma}{\Pi})}T = (\nabla_X T)(X_2, X_3, ..., X_{k+1}) - \sum_{\sigma} \overset{\sigma}{\Pi}(X_{\sigma(1)})T(X_{\sigma(2)}, X_{\sigma(3)}, ..., X_{\sigma(k+1)}),$$

$\overset{\sigma}{\Pi}$ *are 1-forms,* $T \in \mathcal{T}^0_k$ *and the sum includes all permutations* σ *over the set* $(1, 2, ..., k+1)$. *A semi-Riemannian manifold* M *is said to be weakly* T*-symmetric of type-I if* $W^1_{(X,\overset{\sigma}{\Pi})}T = 0$, *for all* $X \in \chi(M)$ *and some 1-forms* $\overset{\sigma}{\Pi}$, *called the associated 1-forms.*

Obviously this operator is of 1st type and non-commutative.

Definition 4.10. *Consider the operator* $W^2_{(X,\Phi,\Pi_i)} : \mathcal{T}^0_k \to \mathcal{T}^0_{k+1}$ *defined by*

$$\begin{aligned}(W^2_{(X,\Phi,\Pi_i)}T)(X_1, X_2, ..., X_k) &= (\nabla_X T)(X_1, X_2, ..., X_k)\\ &- \Phi(X)T(X_1, X_2, ..., X_k) - \sum_{i=1}^{k} \Pi_i(X_i)T(X_1, X_2, ..., \underset{i-th\ place}{X}, ..., X_k),\end{aligned}$$

where Φ *and* Π_i *are 1-forms and* $T \in \mathcal{T}^0_k$. *A semi-Riemannian manifold* M *is said to be weakly* T*-symmetric of type-II if* $W^2_{(X,\Phi,\Pi_i)}T = 0$, *for all* $X \in \chi(M)$ *and some 1-forms* Φ *and* Π_i, *called the associated 1-forms.*

Obviously this operator is of 1st type and non-commutative.

Definition 4.11. *Consider the operator* $W^3_{(X,\Phi,\Pi)} : \mathcal{T}^0_k \to \mathcal{T}^0_{k+1}$ *defined by*

$$W^3_{(X,\Phi,\Pi)}T = \nabla_X T - \Phi \otimes T - \pi_X.T,$$

where Φ *and* Π *are 1-forms and* $T \in \mathcal{T}^0_k$. *A semi-Riemannian manifold* M *is said to be weakly* T*-symmetric of type-III if* $W^3_{(X,\Phi,\Pi)}T = 0$, *for all* $X \in \chi(M)$ *and two 1-forms* Φ *and* Π, *called the associated 1-forms.*

Obviously this operator is of 1st type and non-commutative.

The weak symmetry of type-II was first introduced by Tamássy and Binh [35] and the other two types of the weak symmetry can be deduced from the type-II (see, [17]). Although there are another notion of weak symmetry introduced by Selberg [29] which is totally different from this notion and restriction of this structure is not expressed as an operator on curvature tensor. However, throughout our paper we will consider the weak symmetry in sence of Tamássy and Binh [35].

Definition 4.12. *For a $(0,4)$ tensor D consider the operator $\mathcal{D}(X,Y) : \mathcal{T}^0_r \to \mathcal{T}^0_{r+2}$. A semi-Riemannian manifold is said to be T-semisymmetric type if $\mathcal{D}(X,Y)T = 0$ for all $X, Y \in \chi(M)$. This condition is also written as $D \cdot T = 0$.*

Obviously this operator is of 2nd type and commutative or non-commutative according as D is skew-symmetric or not. Especially if we consider $D = R$, then the manifold is called T-semisymmetric [33].

Definition 4.13. ([1], [9], [10], [14]) *A semi-Riemannian manifold is said to be T-pseudosymmetric type if $(\sum_i c_i D_i) \cdot T = 0$, where $\sum_i c_i D_i$ is a linear combination of $(0,4)$ curvature tensors D_i's over $C^\infty(M)$, $c_i \in C^\infty(M)$.*

Obviously this operator is of 2nd type and generally commutative or non-commutative according as all D_i's are skew-symmetric or not. Consider the special cases $(R - LG) \cdot T = 0$ and $(R - LX \wedge_S Y) \cdot T = 0$. These are known as Deszcz T-pseudosymmetric ([1], [9], [10], [14]) and Ricci generalized T-pseudosymmetric ([6], [7]) respectively. It is clear that the operator of Deszcz pseudosymmetry is commutative but Ricci generalized pseudosymmetry is non-commutative.

5. **Some basic properties of the B-tensor**

In this section we discuss some basic well known properties of the tensor B.

Lemma 5.1. *An operator $\mathcal{L}$ is commutative if $\mathcal{L}g = 0$. Moreover if $\mathcal{L}$ is an endomorphism then this condition is equivalent to the condition that $\mathcal{L}$ is skew-symmetric i.e. $g(\mathcal{L}X, Y) = -g(X, \mathcal{L}Y)$ for all $X, Y \in \chi(M)$.*

Proof: If $\mathcal{L}g = 0$ then, without loss of generality, we may suppose that T is a (0,2) tensor, and then

$$\mathcal{L}(\mathscr{C}(T)) = \mathcal{L}(g^{ij}T_{ij}) = g^{ij}(\mathcal{L}T_{ij}) = \mathscr{C}(\mathcal{L}T),$$

where $\mathscr{C}$ is the contraction operator. Again if $\mathcal{L}$ is an endomorphism then for all $X, Y \in \chi(M)$, $\mathcal{L}g = 0$ implies

$$(\mathcal{L}g)(X,Y) = -g(\mathcal{L}X, Y) - g(X, \mathcal{L}Y) = 0$$

$$\Rightarrow g(\mathcal{L}X, Y) = -g(X, \mathcal{L}Y)$$

$$\Rightarrow \mathcal{L} \text{ is skew-symmetric.}$$

Lemma 5.2. *Contraction and covariant derivative operators are commute each other.*

Lemma 5.3. $Q(g,T) = G \cdot T$.

Proof: For a (0,k) tensor T, we have

$$Q(g,T)(X_1, X_2, \cdots X_k; X, Y) = ((X \wedge_g Y) \cdot T)(X_1, X_2, \cdots X_k).$$

Now $(X \wedge_g Y)(X_1, X_2) = G(X, Y, X_1, X_2)$, so the result follows.

Lemma 5.4. *Let D be a generalized curvature tensor. Then*
(1) $D(X_1, X_2, X_1, X_2) = 0$ implies $D(X_1, X_2, X_3, X_4) = 0$,
(2) $(\mathcal{L}D)(X_1, X_2, X_1, X_2) = 0$ implies $(\mathcal{L}D)(X_1, X_2, X_3, X_4) = 0$, $\mathcal{L}$ is any linear operator.

Proof: The results follows from Lemma 8.9 of [22] and hence we omit it.

We now consider the tensor B and take contraction on i-th and j-th place and get ${}^{ij}S$ for $i, j \in \{1, 2, 3, 4\}$ as

$$(5.1)\quad \begin{cases} {}^{12}S = (-a_1 + a_2 + a_3 + a_5 + a_6 + na_7)S + r(a_4 + a_8 + a_9 + na_{10})g = ({}^{12}p)S + ({}^{12}q)rg \\ {}^{13}S = (-a_0 + a_2 + a_4 + a_5 + na_6 + a_7)S + r(a_3 + a_8 + na_9 + a_{10})g = ({}^{13}p)S + ({}^{13}q)rg \\ {}^{14}S = (a_0 + a_1 + na_2 + a_3 + a_4 + a_6 + a_7)S + r(a_5 + na_8 + a_9 + a_{10})g = ({}^{14}p)S + ({}^{14}q)rg \\ {}^{23}S = (a_0 + a_1 + a_3 + a_4 + na_5 + a_6 + a_7)S + r(a_2 + na_8 + a_9 + a_{10})g = ({}^{23}p)S + ({}^{23}q)rg \\ {}^{24}S = (-a_0 + a_2 + na_3 + a_4 + a_5 + a_7)S + r(a_6 + a_8 + na_9 + a_{10})g = ({}^{24}p)S + ({}^{24}q)rg \\ {}^{34}S = (-a_1 + a_2 + a_3 + na_4 + a_5 + a_6)S + r(a_7 + a_8 + a_9 + na_{10})g = ({}^{34}p)S + ({}^{34}q)rg \end{cases}$$

Again contracting all ${}^{ij}S$ we get ${}^{ij}r$ for $i, j \in \{1, 2, 3, 4\}$ as

$$(5.2)\quad \begin{cases} {}^{12}r = {}^{34}r = (-a_1 + a_2 + a_3 + na_4 + a_5 + a_6 + na_7 + na_8 + na_9 + n^2 a_{10})r \\ {}^{13}r = {}^{24}r = (-a_0 + a_2 + na_3 + a_4 + a_5 + na_6 + a_7 + na_8 + n^2 a_9 + na_{10})r \\ {}^{14}r = {}^{23}r = (a_0 + a_1 + na_2 + a_3 + a_4 + na_5 + a_6 + a_7 + n^2 a_8 + na_9 + na_{10})r \end{cases}$$

Lemma 5.5. *(i) If $S = 0$, then $B = 0$ if and only if $R = 0$.*
(ii) If $\mathcal{L}S = 0$, then $\mathcal{L}B = 0$ if and only if $\mathcal{L}R = 0$, where $\mathcal{L}$ is a commutative 1st type operator and a_i's are constant.
(iii) If $\mathcal{L}S = 0$, then $\mathcal{L}B = 0$ if and only if $\mathcal{L}R = 0$, where $\mathcal{L}$ is a commutative 2nd type operator.

Lemma 5.6. *The tensor B is a generalized curvature tensor if and only if*

$$(5.3)\quad \begin{cases} a_1 = a_4 = a_7 = a_{10} = 0, \\ a_2 = -a_3 = a_5 = -a_6 \text{ and } a_8 = -a_9. \end{cases}$$

Proof: B is a generalized curvature tensor if and only if

$$(5.4)\quad \begin{cases} B(X_1, X_2, X_3, X_4) + B(X_2, X_1, X_3, X_4) = 0, \\ B(X_1, X_2, X_3, X_4) - B(X_3, X_4, X_1, X_2) = 0, \\ B(X_1, X_2, X_3, X_4) + B(X_2, X_3, X_1, X_4) + B(X_3, X_1, X_2, X_4) = 0. \end{cases}$$

Solving the above equations we get the result.

Thus if B is a generalized curvature tensor then B can be written as

$$B = b_0 R + b_1 g \wedge S + b_2 r g \wedge g, \tag{5.5}$$

where b_0, b_1 and b_2 are scalars.

We note that the equation $B(X_1, X_2, X_3, X_4) + B(X_2, X_3, X_1, X_4) + B(X_3, X_1, X_2, X_4) = 0$ can be omitted from the system of equations (5.4) keeping the solution unaltered. Thus the tensor B turns out to be a generalized curvature tensor if and only if

$$B(X_1, X_2, X_3, X_4) + B(X_2, X_1, X_3, X_4) = 0,$$

$$B(X_1, X_2, X_3, X_4) - B(X_3, X_4, X_1, X_2) = 0.$$

Lemma 5.7. *The tensor B is a proper generalized curvature tensor if and only if B is some scalar multiple of R.*

Proof: Let B be a proper generalized curvature tensor. Then B is a generalized curvature tensor also. So B can be written as

$$B = b_0 R + b_1 g \wedge S + b_2 r g \wedge g,$$

where b_0, b_1 and b_2 are scalars. Now $(g \wedge S)$ and $r(g \wedge g)$ both are not proper generalized curvature tensors. Hence for the tensor B to be proper generalized curvature tensor, the scalars b_1 and b_2 must be zero. This proves the result.

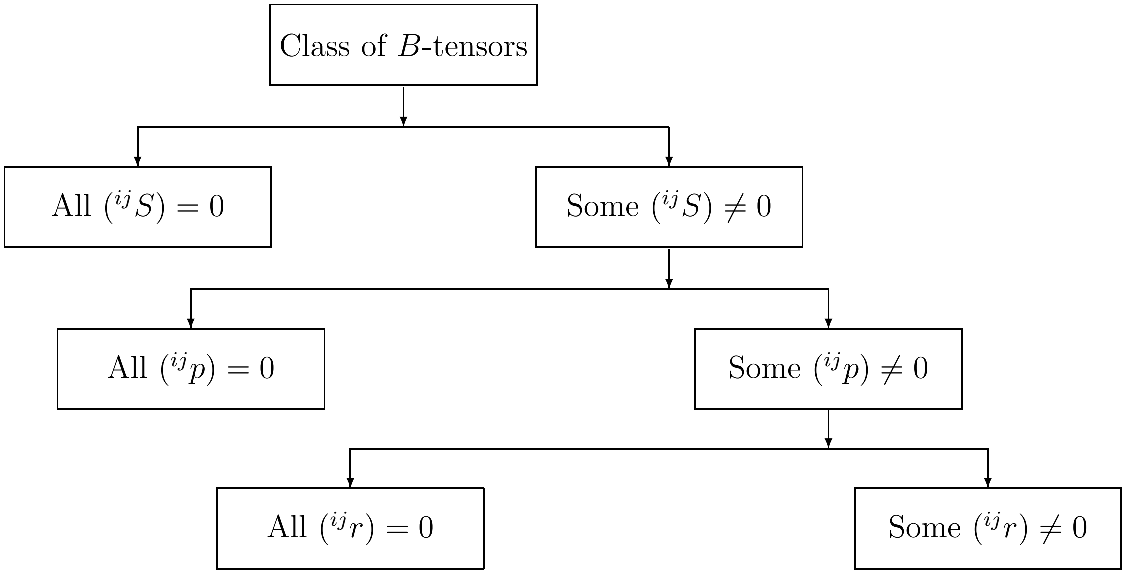
Lemma 5.8. *The endomorphism operator $B(X, Y)$ is skew-symmetric if*

$$a_2 = -a_6, \ a_3 = -a_5, \ a_8 = -a_9, \ a_1 = a_4 = a_7 = a_{10} = 0.$$

Proof: The result follows from solution of the equation $B(X_1, X_2, X_3, X_4) + B(X_1, X_2, X_4, X_3) = 0$.

6. Main Results

In this section we first classify the tensor B with respect to the contraction and then find out the equivalency of some structures for various classes. This classification can express in tree diagram as follows:



Thus we get different classes of B-tensors with respect to contraction given as follows:
(i) **Class 1:** In this class $({}^{ij}S) = 0$ for all $i, j \in \{1, 2, 3, 4\}$. Then we get dependency of a_i's as

$$\left\{\begin{array}{c} a_0 = -a_9(n-2)(n-1),\ a_1 = a_7(n-2), \\ a_2 = a_5 = -a_7 + (n-1)a_9,\ a_3 = a_6 = -(n-1)a_9,\ a_4 = a_7, \\ a_8 = -a_9 + \frac{a_7}{(n-1)},\ a_{10} = -\frac{a_7}{(n-1)}. \end{array}\right. \tag{6.1}$$

An example of such class of B-tensors is conformal curvature tensor C. We take C as the representative member of this class.
(ii) **Class 2:** In this class $({}^{ij}S) \neq 0$ for all $i, j \in \{1, 2, 3, 4\}$ but $({}^{ij}p) = 0$ for all $i, j \in \{1, 2, 3, 4\}$. We get the dependency of a_i's for this class that (6.1) does not satisfy (i.e. one of $a_4 + a_8 + a_9 + na_{10},\ a_3 + a_8 + na_9 + a_{10},\ a_5 + na_8 + a_9 + a_{10},\ a_2 + na_8 + a_9 + a_{10},\ a_6 + a_8 + na_9 + a_{10},\ a_7 + a_8 + a_9 + na_{10}$ is non-zero) but

$$a_0 = a_6(n-2),\ a_1 = a_7(n-2),\ a_2 = a_5 = -a_6 - a_7,\ a_3 = a_6,\ a_4 = a_7. \tag{6.2}$$

An example of such class of B-tensors is conharmonic curvature tensor K. We take K as the representative member of this class.
(iii) **Class 3:** In this class $({}^{ij}p) \neq 0$ for all $i, j \in \{1, 2, 3, 4\}$ but $({}^{ij}r) = 0$ for some $i, j \in \{1, 2, 3, 4\}$. Then for this class a_i's does not satisfy (6.1) but

$$\left\{\begin{array}{c} a_0 = (n-1)(a_3 + a_6 + na_9),\ a_8 = -\frac{a_1 + (n-1)(a_2 + a_3 + a_5 + a_6 + na_9)}{n(n-1)}, \\ a_{10} = \frac{n(a_1 - (n-1)(a_4 + a_7))}{n(n-1)}. \end{array}\right. \tag{6.3}$$

Examples of such class of B-tensors are $W,\ P,\ \mathcal{M},\ P^*,\ \mathcal{W}_0,\ \mathcal{W}_1,\ \mathcal{W}_3^*$. C^* is a member of the class if $a_0 + (n-2)a_2 \neq 0$, otherwise it reduces to conformal curvature tensor. We take W as the representative member of this class. We note that in this case $({}^{ij}p) = n({}^{ij}q)$.

(iv) **Class 4:** $({}^{ij}p)$ and $({}^{ij}r) \neq 0$ for some $i, j \in \{1, 2, 3, 4\}$. For this class a_i's does not satisfy (6.2) and (6.3). Examples of such class of B-tensors are R, $\mathcal{W}_0^*$, $\mathcal{W}_1^*$, $\mathcal{W}_2$, $\mathcal{W}_2^*$, $\mathcal{W}_3$, $\mathcal{W}_4$, $\mathcal{W}_4^*$, $\mathcal{W}_5$, $\mathcal{W}_5^*$, $\mathcal{W}_6$, $\mathcal{W}_6^*$, $\mathcal{W}_7$, $\mathcal{W}_7^*$, $\mathcal{W}_8$, $\mathcal{W}_8^*$, $\mathcal{W}_9$, $\mathcal{W}_9^*$. We take R as the representative member of this class.

We now discuss the equivalency of flatness, symmetry type, recurrency type, semisymmetry type and other various curvature conditions for the above four classes of B-tensors.

Theorem 6.1. *Flatness of all B-tensors of each class are equivalent. Moreover flatness of all B-tensors of a particular class are equivalent to the flatness of the representative member of the class.*

Proof: We first consider the tensor B is of class 1 i.e. satisfying (6.1). We have to show $B = 0$ if and only if $C = 0$. Now

$$\begin{aligned} B_{ijkl} - (a_0 C_{ijkl} + a_1 C_{ikjl}) &= \frac{a_0 \left(-g_{jl}S_{ik} + g_{jk}S_{il} + g_{il}S_{jk} - g_{ik}S_{jl}\right)}{(n-2)} - \frac{a_0 \left(g_{il}g_{jk} - g_{ik}g_{jl}\right) r}{(n-2)(n-1)} \\ &+ \frac{a_1 \left(-g_{kl}S_{ij} + g_{jk}S_{il} + g_{il}S_{jk} - g_{ij}S_{kl}\right)}{n-2} - \frac{a_1 \left(g_{il}g_{jk} - g_{ij}g_{kl}\right) r}{(n-1)(n-2)} \\ &+ a_2 g_{il}S_{jk} + a_3 g_{jl}S_{ik} + a_4 g_{kl}S_{ij} + a_5 g_{jk}S_{il} + a_6 g_{ik}S_{jl} + a_7 g_{ij}S_{kl} \\ &+ r\left(a_8 g_{il}g_{jk} + a_9 g_{ik}g_{jl} + a_{10} g_{ij}g_{kl}\right). \end{aligned} \tag{6.4}$$

As B is of class 1 so simplifying the above using (6.1) we get $B_{ijkl} - (a_0 C_{ijkl} + a_1 C_{ikjl}) = 0$. Again $(a_0 C_{ijkl} + a_1 C_{ikjl}) = 0$ if and only if $C_{ijij} = 0$, i.e. if and only if $C_{ijkl} = 0$ (by Lemma 5.4). Thus we get $B = 0$ if and only if $C = 0$.

Next we consider the tensor B is of class 2 i.e. it satisfies (6.2) but not (6.1). We have to show $B = 0$ if and only if $K = 0$. Now

$$\begin{aligned} B_{ijkl} - (a_0 K_{ijkl} + a_1 K_{ikjl}) &= \frac{a_0 \left(-g_{jl}S_{ik} + g_{jk}S_{il} + g_{il}S_{jk} - g_{ik}S_{jl}\right)}{n-2} \\ &+ \frac{a_1 \left(-g_{kl}S_{ij} + g_{jk}S_{il} + g_{il}S_{jk} - g_{ij}S_{kl}\right)}{n-2} \\ &+ a_2 g_{il}S_{jk} + a_3 g_{jl}S_{ik} + a_4 g_{kl}S_{ij} + a_5 g_{jk}S_{il} + a_6 g_{ik}S_{jl} + a_7 g_{ij}S_{kl} \\ &+ r\left(a_8 g_{il}g_{jk} + a_9 g_{ik}g_{jl} + a_{10} g_{ij}g_{kl}\right). \end{aligned}$$

As B is of class 2 so simplifying the above and using (6.2) we get

$$B_{ijkl} - (a_0 K_{ijkl} + a_1 K_{ikjl}) = r\left(a_8 g_{il}g_{jk} + a_9 g_{ik}g_{jl} + a_{10} g_{ij}g_{kl}\right). \tag{6.5}$$

Now as B and K are both of class 2 so vanishing of any one of B or K implies $r = 0$ and then

$$B_{ijkl} - (a_0 K_{ijkl} + a_1 K_{ikjl}) = 0.$$

Again, $(a_0 K_{ijkl} + a_1 K_{ikjl}) = 0$ if and only if $K_{ijij} = 0$, i.e. if and only if $K_{ijkl} = 0$ (by Lemma 5.4). Thus we get $B = 0$ if and only if $K = 0$.

Again we consider that the tensor B is of class 3 i.e. satisfies (6.3) but not (6.1). We have to show $B = 0$ if and only if $W = 0$. Now

$$\begin{aligned} B_{ijkl} - (a_0 W_{ijkl} + a_1 W_{ikjl}) &= \frac{a_0\left(g_{jk}g_{il} - g_{jl}g_{ik} + g_{il}g_{jk} - g_{ik}g_{jl}\right)}{n(n-i)} \\ &+ \frac{a_1\left(g_{jk}g_{il} - g_{kl}g_{ij} + g_{il}g_{jk} - g_{ij}g_{kl}\right)}{n(n-i)} \\ &+ a_2 g_{il}S_{jk} + a_3 g_{jl}S_{ik} + a_4 g_{kl}S_{ij} + a_5 g_{jk}S_{il} + a_6 g_{ik}S_{jl} + a_7 g_{ij}S_{kl} \\ &+ r\left(a_8 g_{il}g_{jk} + a_9 g_{ik}g_{jl} + a_{10} g_{ij}g_{kl}\right). \end{aligned}$$

As B is of class 3 so simplifying the above and using (6.3) we get

$$\begin{aligned} B_{ijkl} - (a_0 W_{ijkl} + a_1 W_{ikjl}) &= \frac{r}{n}[(a_2 g_{il}g_{jk} + a_3 g_{ik}g_{jl} + a_4 g_{ij}g_{kl} + a_5 g_{il}g_{jk} + a_6 g_{ik}g_{jl} + a_7 g_{ij}g_{kl})] \\ &- [a_2 g_{il}S_{jk} + a_3 g_{jl}S_{ik} + a_4 g_{kl}S_{ij} + a_5 g_{jk}S_{il} + a_6 g_{ik}S_{jl} + a_7 g_{ij}S_{kl}]. \end{aligned} \tag{6.6}$$

Now as B and K are both of class 3 so vanishing of any one of B or W implies $S = \frac{r}{n}g$ and then

$$B_{ijkl} - (a_0 W_{ijkl} + a_1 W_{ikjl}) = 0.$$

Again, $(a_0 W_{ijkl} + a_1 W_{ikjl}) = 0$ if and only if $W_{ijij} = 0$, i.e., if and only if $W_{ijkl} = 0$ (by Lemma 5.4). Thus we get $B = 0$ if and only if $W = 0$.

Finally, we consider that the tensor B is of class 4. We have to show $B = 0$ if and only if $R = 0$. Now as B and R are both of class 4 so vanishing of any one of B or R implies $S = 0$. Then by Lemma 5.5, $B = 0$ if and only if $R = 0$. This completes the proof.

From the proof of the above we can state the following:

Corollary 6.1. *If B is of any class out of the four classes then $R = 0$ implies $B = 0$ and $B = 0$ implies $C = 0$.*

We now discuss the above four classes of B-tensors as equivalence classes of an equivalence relation on the set of all B-tensors $\mathscr{B}$. Consider a relation ρ on $\mathscr{B}$ given by $B_1 \rho B_2$ if and only if B_1-flat (i.e. $B_1 = 0$) $\Leftrightarrow$ B_2-flat (i.e. $B_2 = 0$) , for all $B_1, B_2 \in \mathscr{B}$. It can be easily shown that ρ is an equivalence relation. We conclude from Theorem 6.1 that all B-tensors of class 1 are related to the conformal curvature tensor C, all B-tensors of class 2 are related to the conharmonic curvature tensor K, all B-tensors of class 3 are related to the concircular curvature tensor W, all B-tensors of class 4 are related to the Riemann-Christoffel curvature tensor R. Thus class 1 is the ρ-equivalence class $[C]$, class 2 is the ρ-equivalence class $[K]$, class 3 is the ρ-equivalence class $[W]$ and class 4 is the ρ-equivalence class $[R]$.

Theorem 6.2. (*Characteristic of class* 1) (*i*) *All tensors of class* 1 *are of the form*

$$a_0 C_{ijkl} + a_1 C_{ikjl}$$

and the only generalized curvature tensor of this class is conformal curvature tensor upto a scalar multiple.
(*ii*) *All curvature restrictions of type* 1 *on any* B*-tensor of class* 1 *are equivalent, if* a_0 *and* a_1 *are constant.*
(*iii*) *All curvature restrictions of type* 2 *on any* B*-tensor of class* 1 *are equivalent.*

Proof: We see that if B is of class 1, then from (6.4), $B_{ijkl} = [a_0 C_{ijkl} + a_1 C_{ikjl}]$. Now for B to be a generalized curvature tensor (5.3) fulfilled and we get the form of generalized curvature tensor of this class.
Again applying any restriction $\mathcal{L}B = 0$ implies $\mathcal{L}\,[a_0 C_{ijkl} + a_1 C_{ikjl}] = 0$. Then by Lemma 5.4 we get the result.

Theorem 6.3. (*Characteristic of class* 2) (*i*) *All tensors of class* 2 *are of the form*

$$a_0 K_{ijkl} + a_1 K_{ikjl} + r\left(a_8 g_{il} g_{jk} + a_9 g_{ik} g_{jl} + a_{10} g_{ij} g_{kl}\right)$$

such that $a_8 = \frac{a0+a1}{(n-2)(n-1)}, a_9 = -\frac{a0}{(n-2)(n-1)}, a_{10} = -\frac{a1}{(n-2)(n-1)}$ *does not satisfy all together, otherwise it becomes of class* 1*. The generalized curvature tensor of this class are of the form* $a_0 K + a_8 rG$*,* $a_8 \neq \frac{a0}{(n-1)(n-2)}$.
(*ii*) *All commutative curvature restrictions of type* 1 *on any* B*-tensor of class* 2 *are equivalent, if* a_0*,* a_1*,* a_8*,* a_9 *and* a_{10} *are constant.*
(*iii*) *All commutative curvature restrictions of type* 2 *on any* B*-tensor of class* 2 *are equivalent.*

Proof: We see that if B is of class 2, then from (6.5),

$$B_{ijkl} = a_0 K_{ijkl} + a_1 K_{ikjl} + r\left(a_8 g_{il} g_{jk} + a_9 g_{ik} g_{jl} + a_{10} g_{ij} g_{kl}\right).$$

Now for B to be generalized curvature tensor (5.3) fulfilled and we get the form of generalized curvature tensor of this class as required.
Again applying any commutative operator $\mathcal{L}$ on B implies

$$\mathcal{L}\left[a_0 K_{ijkl} + a_1 K_{ikjl} + r\left(a_8 g_{il} g_{jk} + a_9 g_{ik} g_{jl} + a_{10} g_{ij} g_{kl}\right)\right] = 0.$$

Then by Lemma 5.4 we get the result.

Theorem 6.4. (*Characteristic of class* 3) (*i*) *All tensors of class* 3 *are of the form*

$$(a_0 W_{ijkl} + a_1 W_{ikjl}) + [a_2 g_{il} S_{jk} + a_3 g_{jl} S_{ik} + a_4 g_{kl} S_{ij} + a_5 g_{jk} S_{il} + a_6 g_{ik} S_{jl} + a_7 g_{ij} S_{kl}]$$
$$-\frac{r}{n}\left[(a_2 + a_5) g_{il} g_{jk} + (a_3 + a_6) g_{jl} g_{ik} + (a_4 + a_7) g_{kl} g_{ij}\right]$$

such that $a_2 = a_5 = -\frac{a0+a1}{n-2}, a_3 = a_6 = \frac{a0}{n-2}, a_4 = a_7 = \frac{a1}{n-2}$ *does not satisfy all together, otherwise it becomes of class* 1. *The generalized curvature tensor of this class are of the form* $a_0 W + a_2\left[g \wedge S - \frac{r}{n} G\right]$, $a_2 \neq \frac{a0}{(n-1)(n-2)}$.
(*ii*) *All commutative curvature restrictions of type* 1 *on any* B*-tensor of class* 3 *are equivalent, if* a_0, a_1, a_2, a_3, a_4, a_5, a_6 *and* a_7 *are constant.*
(*iii*) *All commutative curvature restrictions of type* 2 *on any* B*-tensor of class* 3 *are equivalent.*

Proof: The proof is similar to the proof of the Theorem 6.3.

Theorem 6.5. (*Characteristic of class* 4) (*i*) *All commutative curvature restrictions of type* 1 *on any* B*-tensor of class* 4 *are equivalent, if* a_i*'s are all constant.*
(*ii*) *All commutative curvature restrictions of type* 2 *on any* B*-tensor of class* 4 *are equivalent.*

Proof: Consider a commutative operator $\mathcal{L}$ and B is of class 1, such that $\mathcal{L}B = 0$. Now if $\mathcal{L}$ is of 1st type and commutative, then taking contraction we get $\mathcal{L}S = 0$ and $\mathcal{L}(r) = 0$ as a_i's are all constant. Putting this in the expression of $\mathcal{L}B$ we get $\mathcal{L}R = 0$. Again if $\mathcal{L}$ is of 2nd type and commutative, then contraction yields $\mathcal{L}S = 0$ and $\mathcal{L}(r) = 0$. Substituting this in the expression of $\mathcal{L}B$ we get $\mathcal{L}R = 0$. This complete the proof.

We now state some results on particular cases.

Lemma 6.1. [28] *Locally symmetric and projectively symmetric semi-Riemannian manifolds are equivalent.*

Lemma 6.2. [23] *Every concircularly recurrent manifold is necessarily a recurrent manifold with the same recurrence form.*

Lemma 6.3. *Every projectively recurrent manifold is necessarily a recurrent manifold with the same recurrence form.*

From the above four Characteristic theorems of the classes and the Lemma 6.1, 6.2 and 6.3 we can state the results for 1st type operator such that in the following table all condition(s) in a block are equivalent.

Classs 1	**Class 2**	**Class3**	**Class4**
$C = 0$	$K = 0$	$W = 0,\ P = 0,\ \mathcal{M} = 0,$ $P^* = 0,\ \mathcal{W}_0 = 0,$ $\mathcal{W}_1 = 0,\ \mathcal{W}_3^* = 0$	$R = 0,\ \mathcal{W}_0^* = 0,\ \mathcal{W}_1^* = 0,\ \mathcal{W}_2 = 0,$ $\mathcal{W}_2^* = 0,\ \mathcal{W}_3 = 0,$ $\mathcal{W}_i = 0,\ \mathcal{W}_i^* = 0,$ for all $i = 4, 5, \cdots 9$
$\nabla C = 0$	$\nabla K = 0$	$\nabla W = 0,\ \nabla P = 0,\ \mathcal{M} = 0,$ $\nabla P^* = 0,\ \nabla\mathcal{W}_0 = 0,$ $\nabla\mathcal{W}_1 = 0,\ \nabla\mathcal{W}_3^* = 0,$	$\nabla R = 0,\ \nabla\mathcal{W}_0^* = 0,\ \nabla\mathcal{W}_1^* = 0,\ \nabla\mathcal{W}_2 = 0,$ $\nabla\mathcal{W}_2^* = 0,\ \nabla\mathcal{W}_3 = 0,$ $\nabla\mathcal{W}_i = 0,\ \nabla\mathcal{W}_i^* = 0,$ for all $i = 4, 5, \cdots 9$
$\kappa C = 0$	$\kappa K = 0$	$\kappa W = 0,\ \kappa P = 0,\ \kappa\mathcal{M} = 0,$ $\kappa P^* = 0,\ \kappa\mathcal{W}_0 = 0,$ $\kappa\mathcal{W}_1 = 0,\ \kappa\mathcal{W}_3^* = 0,$	$\kappa R = 0,\ \kappa\mathcal{W}_0^* = 0,\ \kappa\mathcal{W}_1^* = 0,\ \kappa\mathcal{W}_2 = 0,$ $\kappa\mathcal{W}_2^* = 0,\ \kappa\mathcal{W}_3 = 0,$ $\kappa\mathcal{W}_i = 0,\ \kappa\mathcal{W}_i^* = 0,$ for all $i = 4, 5, \cdots 9$
$L^sC = 0$	$L^sK = 0$	$L^sW = 0,\ L^sP = 0,$ $L^s\mathcal{M} = 0,\ L^sP^* = 0,$ $L^s\mathcal{W}_0 = 0,\ L^s\mathcal{W}_1 = 0,$ $L^s\mathcal{W}_3^* = 0$	$L^sR = 0,\ L^s\mathcal{W}_0^* = 0,\ L^s\mathcal{W}_1^* = 0,$ $L^s\mathcal{W}_2 = 0,\ L^s\mathcal{W}_2^* = 0,\ L^s\mathcal{W}_3 = 0,$ $L^s\mathcal{W}_i = 0,\ L^s\mathcal{W}_i^* = 0,$ for all $i = 4, 5, \cdots 9$
$\kappa^sC = 0$	$\kappa^sK = 0$	$\kappa^sW = 0,\ \kappa^sP = 0,$ $\kappa^s\mathcal{M} = 0,\ \kappa^sP^* = 0,$ $\kappa^s\mathcal{W}_0 = 0,\ \kappa^s\mathcal{W}_1 = 0,$ $\kappa^s\mathcal{W}_3^* = 0$	$\kappa^sR = 0,\ \kappa^s\mathcal{W}_0^* = 0,\ \kappa^s\mathcal{W}_1^* = 0,$ $\kappa^s\mathcal{W}_2 = 0,\ \kappa^s\mathcal{W}_2^* = 0,\ \kappa^s\mathcal{W}_3 = 0,$ $\kappa^s\mathcal{W}_i = 0,\ \kappa^s\mathcal{W}_i^* = 0,$ for all $i = 4, 5, \cdots 9$

Theorem 6.6. *Let $\mathcal{L}$ be a skew-symmetric operator of type 2, Then the following holds:*
(i) For any two B-tensor B_1 and B_2 of class 1 and 2 respectively, the conditions $\mathcal{L}B_1 = 0$ and $\mathcal{L}B_2 = 0$ are equivalent.
(ii) For any two B-tensor B_1 and B_2 of class 3 and 4 respectively, the conditions $\mathcal{L}B_1 = 0$ and $\mathcal{L}B_2 = 0$ are equivalent.

Proof: To prove this theorem it is sufficient to show that for a skew symmetric second order operator $\mathcal{L}$, $\mathcal{L}C = \mathcal{L}K$ and $\mathcal{L}W = \mathcal{L}R$. Now first consider C and K. Then for any operator $\mathcal{L}$,

$$\mathcal{L}C = \mathcal{L}K + \mathcal{L}\left(\frac{r}{(n-1)(n-2)}G\right).$$

Thus if $\mathcal{L}$ is skew-symmetric second order operator, then $\mathcal{L}\left(\frac{r}{(n-1)(n-2)}G\right) = 0$ and hence (i) is proved. Again considering R and W, we get

$$\mathcal{L}W = \mathcal{L}R + \mathcal{L}\left(\frac{r}{n(n-1)}G\right).$$

So if $\mathcal{L}$ is skew-symmetric second order operator then $\mathcal{L}\left(\frac{r}{n(n-1)}G\right) = 0$ and hence (ii) is proved.

From the above four Characteristic theorems of the classes and the Theorem 6.6 we can state the results for 2nd type operator such that in the following table all conditions in a block of the table are equivalent.

Class 1 and Class 2	**Class 3 and Class 4**
$R \cdot C = 0$, $R \cdot K = 0$	$R \cdot W = 0$, $R \cdot P = 0$, $R \cdot P^* = 0$, $R \cdot \mathcal{M} = 0$, $R \cdot R = 0$, $R \cdot \mathcal{W}_i = 0$, $R \cdot \mathcal{W}_i^* = 0$, for all $i = 0, 1, \cdots 9$
$C \cdot C = 0$, $C \cdot K = 0$	$C \cdot W = 0$, $C \cdot P = 0$, $C \cdot P^* = 0$, $C \cdot \mathcal{M} = 0$, $C \cdot R = 0$, $C \cdot \mathcal{W}_i = 0$, $C \cdot \mathcal{W}_i^* = 0$, for all $i = 0, 1, \cdots 9$
$K \cdot C = 0$, $K \cdot K = 0$	$K \cdot W = 0$, $K \cdot P = 0$, $K \cdot P^* = 0$, $K \cdot \mathcal{M} = 0$, $K \cdot R = 0$, $K \cdot \mathcal{W}_i = 0$, $K \cdot \mathcal{W}_i^* = 0$, for all $i = 0, 1, \cdots 9$
$W \cdot C = 0$, $W \cdot K = 0$	$W \cdot W = 0$, $W \cdot P = 0$, $W \cdot P^* = 0$, $W \cdot \mathcal{M} = 0$, $W \cdot R = 0$, $W \cdot \mathcal{W}_i = 0$, $W \cdot \mathcal{W}_i^* = 0$, for all $i = 0, 1, \cdots 9$
$R \cdot C = LQ(g, C)$, $R \cdot K = LQ(g, K)$	$R \cdot W = LQ(g, W)$, $R \cdot P = LQ(g, P)$, $R \cdot P^* = LQ(g, P^*)$, $R \cdot \mathcal{M} = LQ(g, \mathcal{M})$, $R \cdot R = LQ(g, R)$, $R \cdot \mathcal{W}_i = LQ(g, \mathcal{W}_i)$, $R \cdot \mathcal{W}_i^* = LQ(g, \mathcal{W}_i^*)$, for all $i = 0, 1, \cdots 9$
$C \cdot C = LQ(g, C)$, $C \cdot K = LQ(g, K)$	$C \cdot W = LQ(g, W)$, $C \cdot P = LQ(g, P)$, $C \cdot P^* = LQ(g, P^*)$, $C \cdot \mathcal{M} = LQ(g, \mathcal{M})$, $C \cdot R = LQ(g, R)$, $C \cdot \mathcal{W}_i = LQ(g, \mathcal{W}_i)$, $C \cdot \mathcal{W}_i^* = LQ(g, \mathcal{W}_i^*)$, for all $i = 0, 1, \cdots 9$
$W \cdot C = LQ(g, C)$, $W \cdot K = LQ(g, K)$	$W \cdot W = LQ(g, W)$, $W \cdot P = LQ(g, P)$, $W \cdot P^* = LQ(g, P^*)$, $W \cdot \mathcal{M} = LQ(g, \mathcal{M})$, $W \cdot R = LQ(g, R)$, $W \cdot \mathcal{W}_i = LQ(g, \mathcal{W}_i)$, $W \cdot \mathcal{W}_i^* = LQ(g, \mathcal{W}_i^*)$, for all $i = 0, 1, \cdots 9$
$K \cdot C = LQ(g, C)$, $K \cdot K = LQ(g, K)$	$K \cdot W = LQ(g, W)$, $K \cdot P = LQ(g, P)$, $K \cdot P^* = LQ(g, P^*)$, $K \cdot \mathcal{M} = LQ(g, \mathcal{M})$, $K \cdot R = LQ(g, R)$, $K \cdot \mathcal{W}_i = LQ(g, \mathcal{W}_i)$, $K \cdot \mathcal{W}_i^* = LQ(g, \mathcal{W}_i^*)$, for all $i = 0, 1, \cdots 9$

Thus we can state the following:

Corollary 6.2. (1) *The conditions* $R \cdot R = 0$, $R \cdot W = 0$ *and* $R \cdot P = 0$ *are equivalent.*
(2) *The conditions* $C \cdot R = 0$, $C \cdot W = 0$ *and* $C \cdot P = 0$ *are equivalent.*
(3) *The conditions* $W \cdot R = 0$, $W \cdot W = 0$ *and* $W \cdot P = 0$ *are equivalent.*
(4) *The conditions* $K \cdot R = 0$, $K \cdot W = 0$ *and* $K \cdot P = 0$ *are equivalent.*
(5) *The conditions* $R \cdot C = 0$ *and* $R \cdot K = 0$ *are equivalent.*
(6) *The conditions* $C \cdot C = 0$ *and* $C \cdot K = 0$ *are equivalent.*
(7) *The conditions* $W \cdot C = 0$ *and* $W \cdot K = 0$ *are equivalent.*
(8) *The conditions* $K \cdot C = 0$ *and* $K \cdot K = 0$ *are equivalent.*

Corollary 6.3. (1) *The conditions* $R \cdot R = L_1 Q(g, R)$, $R \cdot W = L_1 Q(g, W)$ *and* $R \cdot P = L_1 Q(g, P)$ *are equivalent.*
(2) *The conditions* $C \cdot R = L_2 Q(g, R)$, $C \cdot W = L_2 Q(g, W)$ *and* $C \cdot P = L_2 Q(g, P)$ *are equivalent.*
(3) *The conditions* $W \cdot R = L_3 Q(g, R)$, $W \cdot W = L_3 Q(g, W)$ *and* $W \cdot P = L_3 Q(g, P)$ *are equivalent.*
(4) *The conditions* $K \cdot R = L_4 Q(g, R)$, $K \cdot W = L_4 Q(g, W)$ *and* $K \cdot P = L_4 Q(g, P)$ *are equivalent.*
(5) *The conditions* $R \cdot C = L_5 Q(g, C)$ *and* $R \cdot K = L_5 Q(g, K)$ *are equivalent.*
(6) *The conditions* $C \cdot C = L_6 Q(g, C)$ *and* $C \cdot K = L_6 Q(g, K)$ *are equivalent.*
(7) *The conditions* $W \cdot C = L_7 Q(g, C)$ *and* $W \cdot K = L_7 Q(g, K)$ *are equivalent.*
(8) *The conditions* $K \cdot C = L_8 Q(g, C)$ *and* $K \cdot K = L_8 Q(g, K)$ *are equivalent.*
Here L_i, $(i = 1, 2, \cdots, 8)$ *are scalars.*

We note that here the operator $\mathcal{P}$ for projective curvature tensor is not considered as P is not skew-symmetric i.e. $\mathcal{P}$ is not of type 2.

7. Conclusion

Form the above discussion we see that the set $\mathscr{B}$ of all B-tensors can be partitioned into four equivalence classes $[C]$ or class 1, $[K]$ or class 2, $[W]$or class 3, and $[R]$ or class 4 under the equivalence relation ρ given by $B_1 \rho B_2$ if and only if $B_1 = 0 \Leftrightarrow B_2 = 0$, where $B_1, B_2 \in \mathscr{B}$. We conclude that

(i) study of any curvature restriction (such as symmetric type, recurrent type, super generalized recurrent, semisymmetric type, pseudosymmetric type) on any B-tensor of class 1 is equivalent to the study of such type of curvature restriction on the conformal curvature tensor C. Thus for all such restrictions, each gives only one structure for all B-tensors of class 1.

(ii) study of a symmetric type and recurrent type curvature restrictions on any B-tensor of class 2 with constant a_i's is equivalent to the study of such type of curvature restriction on the conharmonic curvature tensor K. The study of a commutative semisymmetric type and commutative pseudosymmetric type curvature restrictions on any B-tensor of class 2 is equivalent to the study of such type of restrictions on the conformal curvature tensor C. Moreover, each commutative and first type curvature restrictions on any B-tensor of class 2 with constant coefficients give rise only one structure. Also each commutative and second type curvature restrictions on any B-tensor of class 2 give rise the same structure as to C.

(iii) study of a symmetric type and recurrent type curvature restrictions on any B-tensor of class 3 with constant a_i's is equivalent to the study of such type of curvature restriction on the concircular curvature tensor W. Again the studies of locally symmetric, recurrent, commutative semisymmetric type and commutative pseudosymmetric type curvature restrictions on any B-tensor of class 3 are equivalent to the study of such type of restrictions on the Riemann-Chistoffel curvature tensor R. Moreover, each commutative and first type curvature restrictions on any B-tensor of class 3 with constant coefficients give rise only one structure. Also each commutative and second type curvature restrictions on any B-tensor of class 3 give rise the same structure as to R.

(iv) study of a symmetric type and recurrent type curvature restrictions on any B-tensor of class 4 with constant a_i's is equivalent to the study of such type of curvature restriction on the Riemann-Chistoffel curvature tensor R. The study of a commutative semisymmetric type and commutative pseudosymmetric type curvature restrictions on any B-tensor of class 4 is equivalent to the study of such type of restrictions on the conformal curvature tensor R. Moreover, each commutative and first type curvature restrictions on any B-tensor of class 4 with constant coefficients give rise only one structure. Also each commutative and second type curvature restrictions on any B-tensor of class 4

give rise the same structure as to R.

Finally we also conclude that for future study of any kind of curvature restriction (discussed earlier) on various curvature tensor, we have to study such curvature restriction on the tensor B only and as a particular case we can obtained the results for various curvature tensors. We also note that to study various curvature restrictions on the tensor B, we have to consider the form of B as given in (5.5) but not as the form of (2.1).

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Authors' addresses:

Department of Mathematics,
University of Burdwan, Golapbag,
Burdwan-713104,
West Bengal, India

E-mail address: aask2003@yahoo.co.in